ON THE EXISTENCE OF SOLUTIONS OF EQUILIBRIA IN LUBRICATED JOURNAL BEARINGS∗

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Abstract. In this paper we study a system of equations concerning equilibrium positions of journal bearings. The problem consists of two surfaces in relative motion separated by a small distance filled with a lubricant. The shape of the inlet surface is circular, while the other surface has a more general shape. Our result shows the existence of at least one equilibrium by using degree theory.

Key words. Reynolds variational inequality, inverse problem, existence of solutions, degree theory

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1. Introduction. We consider in this paper a lubricated system called a journal bearing consisting of two cylinders in relative motion. An incompressible fluid, the lubricant, is introduced in the narrow space between the cylinders. An exterior force $F = (F_1, F_2) \in \mathbb{R}^2$ is applied on the inner cylinder (shaft) which turns with a given velocity $\omega$.

The wedge between the two cylinders is assumed to satisfy the thin-film hypothesis, so that the pressure (assumed time-independent) does not depend on the normal coordinate to the bodies and obeys the Reynolds equation.

In order to introduce the Reynolds equation we need to describe the geometry and the dynamic of the system. We suppose that the interior cylinder has a circular form of constant radius (assumed 1) which rotates with known velocity. The transversal axis of the shaft is assumed to have only two degrees of freedom in the transversal plane, i.e., parallel to the exterior cylinder (bush) which is fixed and not necessarily of constant radius.

Let us consider $(O, y_1, y_2)$ a reference system in the transversal plane to the cylinders, and suppose that the distance between $O$ and the surface of the bush is larger than the radius of the shaft (see Figure 1.1). We also assume that the representation in polar coordinates $(r, \theta)$ of the bush is given by

\begin{equation}
(1.1) \quad r = 1 + \delta \rho(\theta),
\end{equation}

where $\rho : [0, 2\pi] \to [1, +\infty]$ is a known function and $\delta > 0$ (the clearance) is a small parameter, representing the distance between the two cylinders when $O$ and the center of the shaft coincide.

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Remark 1.1. The particular case $\rho \equiv 1$ corresponds to a circular bush with radius $1 + \delta$ and $O$ the center of the bush.

Let us now denote by $O_s$ the center of the shaft. The position of $O_s$ is given in cartesian coordinates by $(\delta \eta_1, \delta \eta_2)$ and in polar coordinates by $(\delta \eta, \alpha)$, that is,

$$
\eta_1 = \eta \cos \alpha, \\
\eta_2 = \eta \sin \alpha.
$$

It is well known (see, for instance, [5]) that the distance between the two cylinders is given by $\delta h(\theta, \eta, \alpha) + O(\delta^2)$ with

$$
h(\theta, \eta, \alpha) = \rho(\theta) - \eta \cos(\theta - \alpha) = \rho(\theta) - \eta_1 \cos \theta - \eta_2 \sin \theta.
$$

The formulation of the problem is complete when the distance between the surfaces is of order $\delta$ (i.e., $h \in O(1)$) and second order terms are neglected. Admissible forces in that case are of order $\frac{1}{\delta}$ or smaller. If the distance becomes smaller (i.e., $h \ll 1$) or forces are large (larger than $O(\frac{1}{\delta^2})$), second order terms cannot be neglected and the formulation loses its physical meaning.

Now, the problem will be posed in a fixed domain $\Omega = [0, 2\pi] \times [0, 1]$ which parametrizes the space between shaft and bush. In fact the gap is approximated by the following domain given in cylindrical coordinates by $(r, \theta, x)$ with respect to $O_s$:

$$
1 \leq r \leq 1 + \delta h(\theta, \eta, \alpha), \quad \theta \in [0, 2\pi], \quad x \in [0, 1],
$$

where $O_s = (\delta \eta \cos(\alpha), \delta \eta \sin(\alpha))$.

Since the wedge between the two cylinders satisfies the thin-film hypothesis, the pressure of the lubricant fluid (assumed time-independent) does not depend on the
normal coordinate to the bodies and obeys the Reynolds equation (see [9]). We also consider that there is an alimentation region along the circles \( \{ x = 0 \} \) and \( \{ x = 1 \} \), respectively, where the pressure in the fluid equals the atmospheric pressure supposed to be 0 by translation. Then the pressure \( p : (\theta, x) \in \Omega \to \mathbb{R} \) satisfies the following problem written in nondimensional form:

\[
\begin{cases}
\nabla \cdot (h^3 \nabla p) = \frac{\partial h}{\partial \theta} & \text{on } \Omega, \\
p = 2\pi \text{-periodic in } \theta, \\
p = 0 & \text{on } [0, 2\pi] \times \{0\} \cup [0, 2\pi] \times \{1\}.
\end{cases}
\]

In general the solution of (1.4) is not always nonnegative and we must replace (1.4) by the corresponding variational inequality.

In this work we are interested in an equilibrium problem which entails finding the position \((\eta_1, \eta_2)\) of the shaft such that the hydrodynamic force (load) created by the pressure film equilibrates the exterior force \( F \). Thus the problem is formulated as follows:

Find \( p \in K, (\eta_1, \eta_2) \in A \) such that

\[
\int_{\Omega} h^3 \nabla p \cdot \nabla (\varphi - p) \geq \int_{\Omega} h \frac{\partial}{\partial \theta} (\varphi - p) \quad \forall \varphi \in K,
\]

\[
\int_{\Omega} p \cos \theta d\theta dx = F_1,
\]

\[
\int_{\Omega} p \sin \theta d\theta dx = F_2,
\]

where \( h \) is given in (1.3),

\[
K = \{ \varphi \in H^1_0(\Omega) : \varphi \geq 0 \},
\]

and \( A \subset \mathbb{R}^2 \). The set of admissible positions of the shaft (to be defined later) is such that \( h(\theta, \eta, \alpha) > 0 \quad \forall \theta \in [0, 2\pi] \).

As far as we know, very few works can be found in the literature concerning existence of equilibrium in the lubricated devices in spite of a larger number of references to numerical simulations; see, for instance, [3] or [4].

Exact solutions for the case in which the upper surface is an inclined plane of angle \( \theta \) and infinite width have been known for a long time, since the problem becomes one-dimensional and is easily integrated [7].

For more general shapes of the upper surfaces an existence result is obtained in [1] for the equation case and in [2] for the variational inequality in the one-dimensional case.

The content of the paper is as follows: In section 2 we give the main result. In section 3 we recall some elements of the degree theory which will be used. In section 4 some preliminary results are given, and section 5 is devoted to the proof of the main result.
2. Main results and assumptions. For the rest of the paper we assume that
\begin{equation}
\rho \in C^3(\mathbb{R}), \quad \rho, \rho', \text{ and } \rho'' \text{ are } 2\pi\text{-periodic,}
\end{equation}
\begin{equation}
\rho''(\theta) + \rho(\theta) > 0 \quad \forall \theta \in \mathbb{R},
\end{equation}
\begin{equation}
\min_{0 \leq \theta \leq 2\pi} \rho(\theta) = 1.
\end{equation}

We denote the following:
\begin{equation}
\rho_M = \max_{0 \leq \theta \leq 2\pi} \rho(\theta),
\end{equation}
\begin{equation}
m = \min_{0 \leq \theta \leq 2\pi} (\rho''(\theta) + \rho(\theta)) > 0,
\end{equation}
\begin{equation}
M = \max_{0 \leq \theta \leq 2\pi} (\rho''(\theta) + \rho(\theta)) > 0.
\end{equation}

Remark 2.1. All these assumptions are clearly satisfied in the particular case of a circular bush, corresponding to $\rho \equiv 1$.

Remark 2.2. Assumption (2.2) is the most restrictive of the assumptions. It is introduced for technical reasons and guarantees that in the limit case the set where $h = 0$ is a single line.

Let us introduce the function $a(\alpha) : \mathbb{R} \to \mathbb{R}_+$ given by
\begin{equation}
a(\alpha) := \min_{\alpha - \pi/2 < \theta < \alpha + \pi/2} \left\{ \frac{\rho(\theta)}{\cos(\theta - \alpha)} \right\}.
\end{equation}
The fact that
\[ \lim_{\theta \to \alpha \pm \pi/2} \frac{\rho(\theta)}{\cos(\theta - \alpha)} = \infty \]
and the continuity and boundedness of $\rho$ guarantee the existence of at least one minimum.

Now we define the set $A$ by
\[ A = \left\{ (\eta_1, \eta_2) \in \mathbb{R}^2 : 0 \leq \eta < a(\alpha) \right\}, \]
where $(\eta, \alpha)$ are given by (1.2).

It is clear that $h(\theta) > 0$ $\forall \theta \in [0, 2\pi]$ if and only if $(\eta_1, \eta_2) \in A$.

For any fixed $(\eta_1, \eta_2) \in A$, problem (1.5) has been studied by several authors. The existence and uniqueness of solutions can be obtained by using the direct methods in the calculus of variations along with strict convexity of the associated functional. Then (1.5) admits a unique classical solution $p \in K$; see, for instance, Kinderlehrer and Stampacchia [6].

Thus problem (1.5)–(1.7) is equivalent to the following problem:
\begin{equation}
\begin{cases}
\text{Find } (\eta_1, \eta_2) \in A \quad \text{such that} \\
G(\eta_1, \eta_2) = (0, 0),
\end{cases}
\end{equation}
where $G = (G_1, G_2) : A \to \mathbb{R}^2$ is given by
\begin{equation}
G_1(\eta_1, \eta_2) = \int_\Omega p \cos \theta d\theta dx - F_1,
\end{equation}
\begin{equation}
G_2(\eta_1, \eta_2) = \int_\Omega p \sin \theta d\theta dx - F_2,
\end{equation}
where \( p \) (depending on \( \eta_1, \eta_2 \)) is the unique solution of (1.5).

The main result of the paper is presented in the following theorem.

**Theorem 2.1.** Under assumptions (2.1)–(2.3) and for any \( F \in \mathbb{R}^2 \) there exists at least one solution \((\eta_1, \eta_2) \in A\) of (2.6).

### 3. Known results on degree theory.

In order to prove Theorem 2.1 we use the topological degree theory, which is rapidly recalled in the following.

The topological degree for continuous mappings between \( n \)-dimensional Euclidean spaces was first introduced by L. E. J. Brouwer in 1912. We first introduce the definition of degree for \( C^1 \) maps.

Let \( S \) be a bounded open subset of \( \mathbb{R}^n \) and a \( C^1(S) \) function \( f : S \to \mathbb{R}^n \). Let \( y_0 \in \mathbb{R}^n \) such that \( y_0 \not\in f(\partial S) \), and suppose that \( f \in C^1(S) \) and that \( Df(x) \) is invertible for all \( x \in \partial f^{-1}(y_0) \). Then \( f(x) = y_0 \) has either no solutions in \( S \) or a finite number \( r \) of solutions, say \( x_1, x_2, \ldots, x_r \) and \( \det(Df(x_i)) \neq 0 \) for \( i = 1, 2, \ldots, r \).

**Definition 3.1.** We define the degree of \( f \) in \( S \) at \( y_0 \) as follows:

\[
\text{If } r = 0, \text{ then } d(f, S, y_0) := 0; \text{ else }
\]
\[
d(f, S, y_0) := \sum_{i=1}^r \text{sign} (\det(Df(x_i))).
\]

**Remark 3.1.** In the particular case where \( f \) is a linear function, i.e., \( f(x) = Ax \), where \( A \) is an invertible matrix, the general formula for calculating the degree of linear functions is the following:

\[
\text{deg}(Ax, S, 0) = \text{sgn} (\det A) \quad \text{if } 0 \text{ is contained in } S.
\]

The definition of degree can be extended to continuous maps; see [8, Extension Lemma, p. 60]. For the reader’s convenience we state the following result (see, for instance, [8, Corollary 4 to Theorem 1.12, p. 81]) adapted to the finite-dimensional case.

**Theorem 3.1.** Let \( S \) be a bounded open set in \( \mathbb{R}^n \). Let \( f_0 \) and \( f_1 \) be two continuous functions from \( S \) to \( \mathbb{R}^n \). We assume moreover that \( S \) is a star domain with respect to the point \( y_0 \in S \) and that

\[
[f_0(x) - y_0] \cdot [f_1(x) - y_0] > 0 \text{ for any } x \in \partial S.
\]

Then

\[
d(f_0, S, y_0) = d(f_1, S, y_0).
\]

**Remark 3.2.** It is clear that if \( d(f, S, y_0) \neq 0 \), then there exists at least a solution \( x \in S \) of the equation \( f(x) = y_0 \).
we have that
\[
\min_{0 \leq \theta \leq 2\pi} \left\{ \rho \right\}_{\alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}} \left\{ \frac{1}{\cos(\theta - \alpha)} \right\} \leq a(\alpha)
\]
and
\[
a(\alpha) \leq \max_{0 \leq \theta \leq 2\pi} \left\{ \rho \right\}_{\alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}} \left\{ \frac{1}{\cos(\theta - \alpha)} \right\}.
\]
The fact that
\[
\min_{\alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}} \left\{ \rho \right\} \leq a(\alpha) \leq \max_{\alpha - \frac{\pi}{2} < \theta < \alpha + \frac{\pi}{2}} \left\{ \rho \right\},
\]
implies
\[
\min \{ \rho \} \leq a(\alpha) \leq \max \{ \rho \},
\]
and the proof is complete.

**Lemma 4.2.** There exists \( s \) with \( 0 < s < \frac{\pi}{2} \) such that
\[
|\tilde{\theta}_\alpha - \alpha| < s
\]
for any \( \alpha \in \mathbb{R} \) and for any \( \tilde{\theta}_\alpha \in [\alpha - \frac{\pi}{2}, \alpha + \frac{\pi}{2}] \) satisfying
\[
\frac{\rho(\tilde{\theta}_\alpha)}{\cos(\tilde{\theta}_\alpha - \alpha)} = a(\alpha).
\]

*Proof.* From Lemma 4.1 we deduce
\[
\cos(\tilde{\theta}_\alpha - \alpha) \geq \frac{1}{\rho_M},
\]
which implies
\[
|\tilde{\theta}_\alpha - \alpha| \leq \arccos \frac{1}{\rho_M} < \frac{\pi}{2}.
\]
Now taking \( s = \frac{1}{2}(\frac{\pi}{2} + \arccos \frac{1}{\rho_M}) \) we obtain the result.

**Lemma 4.3.** Under assumptions (2.1)–(2.3), for any \( \alpha \in \mathbb{R} \), the set of \( \theta \in [\alpha - \frac{\pi}{2}, \alpha + \frac{\pi}{2}] \), which minimizes the function \( \theta \mapsto \frac{\rho(\theta)}{\cos(\theta - \alpha)} \), is a single point which we denote by \( \theta_\alpha \) satisfying
\[
\frac{\rho(\theta_\alpha)}{\cos(\theta_\alpha - \alpha)} = a(\alpha)
\]
and
\[
\rho'(\theta_\alpha) \cos(\theta_\alpha - \alpha) + \rho(\theta_\alpha) \sin(\theta_\alpha - \alpha) = 0.
\]
Moreover the function \( \alpha \mapsto \theta_\alpha \) belongs to \( C^1(\mathbb{R}) \).

*Proof.* Let \( \alpha \) be a fixed value of \( \mathbb{R} \) and consider the critical points of the function \( \theta \in [\alpha - \frac{\pi}{2}, \alpha + \frac{\pi}{2}] \mapsto \frac{\rho(\theta)}{\cos(\theta - \alpha)} \), which satisfies
\[
\left( \frac{\rho(\theta)}{\cos(\theta - \alpha)} \right)' = \frac{\rho'(\theta)}{\cos(\theta - \alpha)} + \frac{\rho(\theta) \sin(\theta - \alpha)}{\cos(\theta - \alpha)^2} = 0.
\]
Consider
\[ f(\theta, \alpha) := \rho' \cos(\theta - \alpha) + \rho \sin(\theta - \alpha). \]

Notice that if \( \theta \) is a minimum of \( \frac{\rho(\theta)}{\cos(\theta - \alpha)} \), then \( f(\theta, \alpha) = 0 \). We now have that
\[ \frac{\partial f}{\partial \theta} = (\rho'' + \rho) \cos(\theta - \alpha), \]
which is positive in \(|\theta - \alpha| < \frac{\pi}{2}\) (by hypothesis (2.2)). The assertion follows using also the implicit function theorem on the open set \((\theta, \alpha) \in [-s, s] \times \mathbb{R}\) with \(s\) given in Lemma 4.2.

We now introduce the function \( h_0 : \mathbb{R}^2 \to \mathbb{R} \) defined by
\[ h_0(\theta, \alpha) = h(\theta, a(\alpha), \alpha) = \rho(\theta) - a(\alpha) \cos(\theta - \alpha). \]

**Remark 4.1.** We have from Lemma 4.3
\[ h_0(\theta, \alpha) = \frac{\partial h_0}{\partial \theta}(\theta, \alpha) = 0, \]
\[ \frac{\partial^2 h_0}{\partial \theta^2}(\theta, \alpha) = \rho''(\theta, \alpha) + \rho(\theta, \alpha). \]

**Lemma 4.4.** Let us denote for any \( \alpha \in [0, 2\pi] \) and \( \epsilon > 0 \) small enough
\[ I_{\alpha, \epsilon} := [\theta_{\alpha} - 2\sqrt{\epsilon}, \theta_{\alpha} - \sqrt{\epsilon}]. \]
Then for any \( \alpha \in [0, 2\pi] \) and \( \theta \in I_{\alpha, \epsilon} \) we have
(i) \( h(\theta, a(\alpha) - \epsilon, \alpha) \leq (2M + 2)\epsilon \),
(ii) \( -\frac{\partial h}{\partial \theta}(\theta, a(\alpha) - \epsilon, \alpha) \geq \frac{m}{2} \sqrt{\epsilon} \),
with \( m \) and \( M \) defined in (2.4).

**Proof.**
(i) We have
\[ h(\theta, a(\alpha) - \epsilon, \alpha) = h_0(\theta, \alpha) + \epsilon \cos(\theta - \alpha). \]
From Taylor development of \( h_0 \) at \( \theta = \theta_{\alpha} \), using (4.3) and (4.4) we have (i).
(ii) We have
\[ -\frac{\partial h}{\partial \theta}(\theta, a(\alpha) - \epsilon, \alpha) = -\frac{\partial h_0}{\partial \theta}(\theta, \alpha) + \epsilon \sin(\theta - \alpha). \]
Using the Taylor development of \( \frac{\partial h_0}{\partial \theta} \) at \( \theta = \theta_{\alpha} \) we obtain
\[ -\frac{\partial h}{\partial \theta}(\theta, a(\alpha) - \epsilon, \alpha) = \frac{\partial^2 h_0}{\partial \theta^2}(\theta_{\alpha} - \theta) + \epsilon \sin(\theta - \alpha) \]
with \( \hat{\theta} \in I_{\alpha, \epsilon} \).
From (4.4), the uniform continuity of \( \frac{\partial^2 h_0}{\partial \vartheta^2} \), and (2.4) we deduce
\[
\frac{\partial^2 h_0}{\partial \vartheta^2}(\hat{\theta}, \alpha) \geq \frac{2}{3}m.
\]
Since \( \theta - \alpha \geq \sqrt{\epsilon} \), we obtain for \( \epsilon \) small enough
\[
-\frac{\partial h}{\partial \vartheta}(\theta, a(\alpha) - \epsilon, \alpha) \geq \frac{2}{3}m\sqrt{\epsilon} - \epsilon,
\]
which ends the proof. \( \square \)

**Lemma 4.5.** Let \( \epsilon \) be small enough and \( \eta := a(\alpha) - \epsilon \). There exists a constant \( c > 0 \) independent of \( \epsilon \) and \( \alpha \) such that
\[
\inf_{0 \leq \alpha \leq 2\pi} \int_{\Omega} h^3(\theta, \eta, \alpha) |\nabla p|^2 d\theta dx > c\epsilon^{-\frac{1}{2}} \quad \text{as } \epsilon \to 0.
\]

**Proof.** We take in (1.5) \( \phi := p + \varphi \) with \( \varphi \in K \) arbitrary. Then
\[
\int_{\Omega} h \frac{\partial \varphi}{\partial \vartheta} d\theta dx \leq \int_{\Omega} h^3 \nabla p \nabla \varphi d\theta dx.
\]
By the Cauchy–Schwarz inequality we have
\[
\int_{\Omega} h \frac{\partial \varphi}{\partial \vartheta} d\theta dx \leq \left| \int_{\Omega} h^3 |\nabla p|^2 d\theta dx \right|^{\frac{1}{2}} \left| \int_{\Omega} h^3 |\nabla \varphi|^2 d\theta dx \right|^{\frac{1}{2}}
\]
and we deduce
\[
\left(4.5\right) \quad \left| \int_{\Omega} h^3 |\nabla p|^2 d\theta dx \right|^{\frac{1}{2}} \geq \sup_{\varphi \in K, \varphi \neq 0} \frac{-\int_{\Omega} \varphi \frac{\partial h}{\partial \vartheta} d\theta dx}{\int_{\Omega} h^3 |\nabla \varphi|^2 d\theta dx}.
\]
Let \( m \) be given by (2.4) and \( \psi \in C^2(\mathbb{R}) \) such that
(i) \( \text{supp}(\psi) \subset [-2, -1] \);
(ii) \( \psi \geq 0 \);
(iii) \( \int_{\mathbb{R}} \psi > 0 \).
As a consequence of (i)–(iii) we have
\[
\left(4.6\right) \quad \frac{\int_{\mathbb{R}} \psi}{|\int_{\mathbb{R}} |\psi'|^2|^{\frac{1}{2}}} = c_1 > 0.
\]
Let
\[
\varphi_{\epsilon}(\theta, x) := \psi \left( \frac{\theta - \theta_{\alpha}}{\sqrt{\epsilon}} \right) x(1-x)
\]
with \( \theta_{\alpha} \) defined in Lemma 4.3.
It is clear that \( \text{supp}(\varphi_{\epsilon}) = I_{\alpha, \epsilon} \times [0, 1] \) with \( I_{\alpha, \epsilon} \) as in Lemma 4.4. Using Lemma 4.4 (ii) we deduce
\[
\left(4.7\right) \quad -\int_{\Omega} \frac{\partial h}{\partial \vartheta} \varphi_{\epsilon} d\theta dx \geq \min_{\theta \in I_{\alpha, \epsilon}} \left( \frac{\partial h}{\partial \vartheta} \right) \int_{\Omega} \varphi_{\epsilon} d\theta dx \geq \frac{m}{2} \int_{\mathbb{R}} \psi dy \int_{0}^{1} x(1-x) dx = \frac{m}{2} \int_{\mathbb{R}} \psi dy.
\]
On the other hand, using Lemma 4.4 (i) we have
\[ \left| \int_{\Omega} h^3 |\nabla \varphi|^2 d\theta dx \right|^{\frac{1}{2}} \leq \max_{i, \alpha} \left\{ h^3 \right\} \frac{1}{2} \left| \int_{\Omega} |\nabla \varphi|^2 d\theta dx \right|^{\frac{1}{2}} \]
\[ \leq ((2M + 2)\epsilon)^{3/2} \left| \int_{\Omega} |\nabla \varphi|^2 d\theta dx \right|^{\frac{1}{2}}. \]

A simple calculation gives
\[ \int_{\Omega} |\nabla \varphi| d\theta dx \leq \epsilon^{-1/2}. \]

We then have
\[ \left| \int_{\Omega} h^3 |\nabla \varphi|^2 d\theta dx \right|^{\frac{1}{2}} \leq c_4 \epsilon^{5/4}, \]
which, also using (4.7), gives us
\[ \left| - \int_{\Omega} \frac{\partial h}{\partial \theta} d\theta dx \right| \left| \frac{1}{\int_{\Omega} h^3 |\nabla \varphi|^2 d\theta dx} \right|^{1/2} \geq c_5 \epsilon^{-\frac{3}{4}}. \]

Finally from (4.5) and (4.9) we have the result.

We now introduce for any \( \epsilon \) small the function \( \tilde{h} : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by
\[ \tilde{h}(\theta, \alpha) := h(\theta, a(\alpha) - \epsilon, \alpha) + [\rho''(\theta) + \rho(\theta)](1 - \cos(\theta - \theta_\alpha)). \]

Notice that
\[ \left\{ \begin{array}{l}
    h(\theta, a(\alpha) - \epsilon, \alpha) = \tilde{h}(\theta, \alpha) = \epsilon \cos(\theta_\alpha - \alpha), \\
    \frac{\partial h}{\partial \theta}(\theta, a(\alpha) - \epsilon, \alpha) - \frac{\partial h}{\partial \theta}(\theta, a(\alpha) - \epsilon, \alpha) = -\epsilon \sin(\theta_\alpha - \alpha), \\
    \frac{\partial^2 h}{\partial \theta^2}(\theta, a(\alpha) - \epsilon, \alpha) - \frac{\partial^2 h}{\partial \theta^2}(\theta, a(\alpha) - \epsilon, \alpha) = -\epsilon \cos(\theta_\alpha - \alpha).
\end{array} \right. \]

**Lemma 4.6.** There exists a constant \( c \) independent of \( \epsilon \) and \( \alpha \) such that
\[ \int_{0}^{2\pi} \frac{\left( h(\theta, a(\alpha) - \epsilon, \alpha) - \tilde{h}(\theta, \alpha) \right)^2}{h^3(\theta, a(\alpha) - \epsilon, \alpha)} d\theta \leq c. \]

**Proof.** By the Taylor polynomial of \( h - \tilde{h} \) and \( h \) at \( \theta_\alpha \) and (4.11) we have
\[ |h(\theta, a(\alpha) - \epsilon, \alpha) - \tilde{h}(\theta, \alpha)| \leq \epsilon|\theta - \theta_\alpha| + \epsilon|\theta - \theta_\alpha|^2 + c|\theta - \theta_\alpha|^3 \]
and
\[ h(\theta, a(\alpha) - \epsilon, \alpha) = \epsilon \cos(\theta_\alpha - \alpha) - \epsilon \sin(\theta_\alpha - \alpha)(\theta - \theta_\alpha) + \frac{1}{2} \left( \rho''(\theta) + \rho(\theta) - \epsilon \cos(\theta_\alpha - \alpha) \right)(\theta - \theta_\alpha)^2 \\
+ \frac{1}{6} \frac{\partial^3 h}{\partial \theta^3}(\hat{\theta}, a(\alpha) - \epsilon, \alpha)(\theta - \theta_\alpha)^3 \]
with \( \hat{\theta} \in \mathbb{R} \).
From Lemma 4.2, (4.13), and (2.2) there exists $m_1, m_2$, and $m_3$ positive and independent of $\epsilon$ and $\alpha$ such that

\begin{equation}
(4.14) \quad h(\theta, a(\alpha) - \epsilon, \alpha) \geq m_1 \epsilon + m_2 (\theta - \theta_\alpha)^2
\end{equation}

for all $\theta$ such that $|\theta - \theta_\alpha| \leq m_3$.

We have from (4.12)

\[ \int_0^{2\pi} \frac{(h - \tilde{h})^2}{h^3} d\theta \leq 2(\epsilon^2 I_1 + \epsilon^2 I_2 + c^2 I_3) \]

with $I_k = \int_0^{2\pi} \frac{(a - \theta_\alpha)^k}{h^k} d\theta$, $k = 1, 2, 3$.

Now from (4.14) it is clear that $I_3$ is bounded uniformly in $\epsilon$ and $\alpha$.

We will prove the uniform estimate for $\epsilon^2 I_1$. For $\epsilon^2 I_2$ the proof is similar.

We have

\[ I_1 = I_1^1 + I_1^2 + I_1^3, \]

where $I_1^1, I_1^2, I_1^3$ are the subintegrals, respectively, in the intervals $|\theta - \theta_\alpha| \geq m_3, \epsilon^{1/3} \leq |\theta - \theta_\alpha| \leq m_3$, and $0 \leq |\theta - \theta_\alpha| \leq \epsilon^{1/3}$.

It is clear that $\epsilon^2 I_1^1$ is bounded uniformly in $\epsilon$ and $\alpha$ since $h$ is lower bounded by a positive constant on the interval $|\theta - \theta_\alpha| \geq m_3$.

From (4.14) we have

\[ I_1^2 \leq m_2^{-3} \int_{\epsilon^{1/3} \leq |\theta - \theta_\alpha| \leq m_3} \frac{d\theta}{|\theta - \theta_\alpha|^4} \leq c \epsilon^{-4/3} \]

and

\[ I_1^3 \leq m_1^{-3} \epsilon^{-3} \int_{0 \leq |\theta - \theta_\alpha| \leq \epsilon^{1/3}} |\theta - \theta_\alpha|^2 d\theta \leq m_1^{-3} \epsilon^{-2}, \]

which proves the lemma.

5. Proof of Theorem 2.1. We apply the degree theory recalled in section 3 to the function $G$.

Since $G$ is not defined on $\bar{A}$ we introduce for any $\epsilon > 0$ small enough the domain

\[ A_\epsilon = \{(\eta_1, \eta_2) : 0 \leq \eta \leq a(\alpha) - \epsilon\}. \]

Let us now introduce the vector field

\[ W : (\eta_1, \eta_2) \in A \rightarrow (\eta \sin \theta_\alpha, -\eta \cos \theta_\alpha) \in \mathbb{R}^2 \]

with $(\eta, \alpha)$ defined in (1.2) and $\theta_\alpha$ as in Lemma 4.3.

We observe that

\[ W \cdot (-\eta_2, \eta_1) = -\eta^2 \cos(\theta_\alpha - \alpha). \]

From Lemmas 4.2 and 4.3 we have that

\[ W \cdot (-\eta_2, \eta_1) < 0 \quad \forall (\eta_1, \eta_2) \in A. \]
From Theorem 3.1 we deduce that
\[ \deg(W, A, 0) = \deg((-\eta_2, \eta_1), A, 0) = 1. \]

Notice that \( \deg(f, S, 0) = (-1)^n \deg(-f, S, 0) \), where \( n \) is the dimension of the domain (for the application here, \( n = 2 \), so the sign does not change).

It suffices now to prove the inequality
\[ G(\eta_1, \eta_2) \cdot W(\eta_1, \eta_2) > 0 \quad \forall (\eta_1, \eta_2) \in \partial A, \]
and the proof is finished using again Theorem 3.1.

We have, \( \forall (\eta_1, \eta_2) \in \partial A, \eta = a(\alpha) - \epsilon. \) Then
\[ G(\eta_1, \eta_2) \cdot W(\eta_1, \eta_2) = -\eta \int_{\Omega} p \sin(\theta - \theta_\alpha) d\theta dx + \eta(-F_1 \sin \theta_\alpha + F_2 \cos \theta_\alpha). \]

Now taking \( \varphi = 0 \) and \( \varphi = 2\rho \), respectively, in (1.5) we obtain
\[ \int_{\Omega} h^3 |\nabla p|^2 d\theta dx = -\int_{\Omega} \frac{\partial h}{\partial \theta} p d\theta dx. \]
Notice that
\[ \int_{\Omega} \frac{\partial h}{\partial \theta} p d\theta dx = \int_{\Omega} \frac{\partial}{\partial \theta} (h - \tilde{h}) p d\theta dx + \int_{\Omega} \frac{\partial \tilde{h}}{\partial \theta} p d\theta dx \]
with \( \tilde{h} \) given by (4.10).

We now have
\[ \int_{\Omega} \frac{\partial}{\partial \theta} (h - \tilde{h}) p d\theta dx = -\int_{\Omega} (h - \tilde{h}) \frac{\partial p}{\partial \theta} d\theta dx \]
\[ \geq -\int_{\Omega} \frac{|h - \tilde{h}|}{h^{3/2}} h^{3/2} |\nabla p| d\theta dx \]
\[ \geq -\frac{1}{2} \int_{\Omega} \frac{|h - \tilde{h}|^2}{h^3} d\theta dx - \frac{1}{2} \int_{\Omega} h^3 |\nabla p|^2 d\theta dx. \]

Since
\[ \frac{\partial h}{\partial \theta} = (\rho''(\theta_\alpha) + \rho(\theta_\alpha)) \sin(\theta - \theta_\alpha) \]
we obtain from (5.3) and (5.5)
\[ \int_{\Omega} h^3 |\nabla p|^2 d\theta dx \leq \frac{1}{2} \int_{\Omega} \frac{|h - \tilde{h}|^2}{h^3} d\theta dx + \frac{1}{2} \int_{\Omega} h^3 |\nabla p|^2 d\theta dx \]
\[ - (\rho''(\theta_\alpha) + \rho(\theta_\alpha)) \int_{\Omega} \sin(\theta - \theta_\alpha) p d\theta dx, \]
which implies, using also hypotheses (2.2),
\[ - \int_{\Omega} \sin(\theta - \theta_\alpha) p d\theta dx \geq \frac{\Omega h^3 |\nabla p|^2 d\theta dx - \int_{\Omega} \frac{|h - \tilde{h}|^2}{h^3} d\theta dx}{2(\rho''(\theta_\alpha) + \rho(\theta_\alpha))}. \]
Since $\rho$ is in $C^2$ and from Lemmas 4.5 and 4.6, we obtain for $\epsilon$ small enough

$$- \int_{\Omega} \sin(\theta - \theta_{\alpha}) p \geq c(\epsilon^{-1/2} - 1)$$

with $c > 0$ a constant independent of $\epsilon$ and $\alpha$.

Since $\eta = a(\alpha) - \epsilon$ and thanks to (5.2) we deduce

$$G(\eta_1, \eta_2) \cdot W(\eta_1, \eta_2) \geq \eta(c(\epsilon^{-1/2} - 1) - \|F\|).$$

Taking $\epsilon$ small enough, we prove the theorem.

Remark 5.1. In the same manner we can prove the existence of at least one equilibrium solution for some other similar problems in this context. For instance, the case of Dirichlet boundary conditions on every boundary of $\Omega$, or the case of a one-dimensional domain with Dirichlet boundary conditions.

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